RESONANCE BEHAVIOR OF A DYNAMICAL SYSTEM WITH COLLISIONS

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In this article forced vibrations of a two-mass dynamical system with collisions are investigated. The simplest periodic motion of the system under consideration has been studied previously in a number of papers [1 to 3]. It has been established that, in a fairly narrow range of variation of the parameters, collisions lead to a considerable decrease in the vibrations. However, similar nonlinear systems exhibit various types of forced vibrations [4] which can occur in other ranges of values of the parameters and which do not exclude the appearance of significantly different behavior of the system. Therefore, the investigation of the effect of changes of the parameters on the character of the motion is of some interest.

The present problem is solved in this paper by reducing the equations of ions of motion to certain point transformations of four-dimensional surfaces, and solving these on the computer. Since by its very nature a digital computer is, generally speaking, well adapted to carrying out point transformations regardless of the character of the nonlinearities, its use seems to be more effective than solving the system under study with an analogue machine [5].

As a result of the study of the behavior of the phase trajectories (whose structure may become more and more complex as the time of solution increases), a region of values of the parameters is singled out, for which sharply defined resonances occur. Therefore, under some conditions, damping by impact (acceleration damping) can lead to a pronounced increase in the vibrations of the system.

1. Equations of motion of the system. The model which is used for the investigation is shown in Fig. 1. Mass M on an elastic support is acted upon by the force $F \cos \Omega t$. The motion of the second mass m is limited by two stops. The interaction of the masses takes place only at the instant of impact and when they move together as a single mass. Under the usual assumptions for this type of problem, the equations of motion are written as follows.

For independent motions of the two masses in a time interval between collisions,

$$M\xi^{\cdot\cdot} + k\xi = F \cos \Omega t, \qquad m\eta^{\cdot\cdot} = 0, \qquad |\eta - \xi| < D \tag{1.1}$$

The relations between the velocities of the masses before and after impact are

$$(M + m)\,\xi^{\cdot\prime} = (M - mR)\,\xi^{\cdot} + m\,(1 + R)\,\eta^{\cdot}$$
$$(M + m)\,\eta^{\cdot\prime} = M\,(1 + R)\,\xi^{\cdot} + (m - MR)\,\eta^{\cdot}, \quad |\eta - \xi| = D \tag{1.2}$$

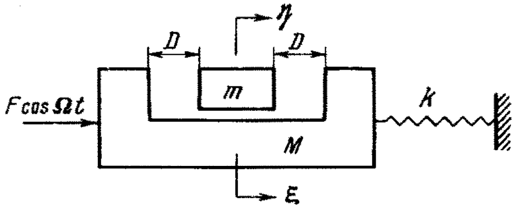

FIG. 1

The motion when the masses move together after a perfectly inelastic impact or after an impact at zero relative velocity, provided that at the instant of impact the inequality

$$(\xi^{\cdot\cdot} - \eta^{\cdot\cdot})(\xi - \eta) > 0$$

is satisfied, is described by

$$(M + m)\,\xi^{\cdot\cdot} + k\xi = F\cos\Omega t$$
$$\xi^{\cdot\cdot}\,(\xi - \eta) > 0, \quad |\eta - \xi| = D \tag{1.3}$$

The position of the mass M which corresponds to the undeformed state of the spring k is taken as the origin $\xi = 0$. The origin $\eta = 0$ corresponds to the position of the mass m, where for $\xi = 0$ the clearances between the mass m and the two stops are both equal to D. It is assumed that the coefficient of restitution R for the impact can vary from zero to one.

The introduction of the dimensionless variables

$$x = \xi k / F, \qquad y = (\eta - \xi)\,k / F, \qquad \tau = t\,\sqrt{k / M} \tag{1.4}$$

permits us to write the equations (1.1) to (1.3) in the simpler form

$$x^{\cdot\cdot} + x = \cos\omega\tau, \qquad y^{\cdot\cdot} + x^{\cdot\cdot} = 0, \qquad |y| < d \tag{1.5}$$

$$y^{\cdot\prime} = -Ry^{\cdot}, \qquad x^{\cdot\prime} = x^{\cdot} + \frac{\mu\,(1 + R)}{1 + \mu}\,y^{\cdot}, \qquad |y| = d \tag{1.6}$$

$$(1 + \mu)\,x^{\cdot\cdot} + x = \cos\omega\tau, \qquad x^{\cdot\cdot}\,y < 0, \qquad |y| = d \tag{1.7}$$

The dimensionless parameters of the system, μ, d, and ω are expressed in terms of the original parameters by the relations

$$\mu = m / M, \qquad d = Dk / F, \qquad \omega = \Omega\,\sqrt{M / k} \tag{1.8}$$

The motion of the system (1.5) to (1.7) takes place in a five-dimensional phase space* x, $x^{\cdot}$, y, $y^{\cdot}$, τ, whose structure depends on four parameters μ, d, ω, and R.

2. Classification of the point transformations. The region of motion of the representative point in phase space is bounded by the surfaces where collisions of the masses occur, $y = +d$ and $y = -d$. It is therefore expedient [4] in the study of the solutions of the system (1.5) to (1.7) to investigate the point transformations of these surfaces. Each phase trajectory can be considered as consisting of parts which are determined by Equations (1.5), (1.6), or (1.7). The collisions described by Equation (1.6) refer to the ends of the parts of a trajectory.

Equations (1.5) and (1.6) define four possible point transformations T_{+-} and T_{-+} for which the original and the transformed points are located on different surfaces $y = +d$ and $y = -d$, and another two transformations T_{++} and T_{--} the original and

* More correctly a state space. (Translator's note)

transformed points of which are located on the same surface.

Equation (1.7) defines two possible point transformations. The transformation S_{++} corresponds to common motion of the masses on the surface $y = +d$, and S_{--} corresponds to common motion on the surface $y = -d$.

The introduction of the six point transformations which have been enumerated permits consideration of any motion of the system under study as the result of a sequence of applications of the transformations. In this connection, the first index of each successive transformation must coincide with the second index of the preceding transformation. Thus, for example, after the transformation T_{-+}, the transformations T_{+-}, T_{++}, or S_{++} may be applied; after the transformation S_{--}, the transformations T_{-+} or T_{--}, etc.

We shall denote the initial point of a transformation by $M_0\{x_0, x_0^{\cdot}, y_0, y_0^{\cdot}, \tau_0\}$ and the final point $M_1\{x_1, x_1^{\cdot}, y_1, y_1^{\cdot}, \tau_1\}$. By solving Equation (1.5) and assuming that at the instant of time τ_1 a collision (1.6) takes place, we obtain the following transformation of T-type in general form:

$$x_1 = (x_0^{\cdot} + \omega\alpha \sin\omega\tau_0)\sin(\tau_1 - \tau_0) + (x_0 - \alpha\cos\omega\tau_0)\cos(\tau_1 - \tau_0) + \alpha\cos\omega\tau_1$$

$$\begin{aligned} x_1^{\cdot} = \frac{1-\mu R}{1+\mu}\,[(x_0^{\cdot} + \omega\alpha\sin\omega\tau_0)\cos(\tau_1 - \tau_0) - (x_0 - \alpha\cos\omega\tau_0)\sin(\tau_1 - \tau_0) - \\ - \omega\alpha\sin\omega\tau_1] + \frac{\mu(1+R)}{1+\mu}(x_0^{\cdot} + y_0^{\cdot}) \end{aligned} \tag{2.1}$$

$$y_1 = y_0 + x_0 + (x_0^{\cdot} + y_0^{\cdot})(\tau_1 - \tau_0) - x_1$$

$$(1 - \mu R)\, y_1^{\cdot} = R\,(1+\mu)(x_1^{\cdot} - x_0^{\cdot} - y_0^{\cdot})$$

where $\alpha = (1 - \omega^2)^{-1}$ and the value $\tau_1 > \tau_0$ and is the smallest root of the equation

$$|\, y_1(\tau_1)\,| = d \tag{2.2}$$

	T_{++}	T_{+-}	T_{-+}	T_{--}
y_0	d	d	$-d$	$-d$
y_1	d	$-d$	d	$-d$

The equations for each of the four transformations of T-type are obtained from the general equations (2.1) and (2.2) by substituting the specific values of y_0 and y_1 into them in accordance with the table given to the left.

We note that the equations for the transformations T_{+-} and T_{-+} permit determination of the motion of the system even in the case where the point M_0 is located between the surfaces representing the collisions, i.e. for $|y_0| < d$.

The transformation equations of S-type are obtained by solving the equation (1.7) in the following form

$$\begin{aligned} x_1 &= \gamma^{-1}(x_0^{\cdot} + \omega\beta\sin\omega\tau_0)\sin\gamma(\tau_1 - \tau_0) + (x_0 - \beta\cos\omega\tau_0)\cos\gamma(\tau_1 - \tau_0) + \beta\cos\omega\tau_1 \\ x_1^{\cdot} &= (x_0^{\cdot} + \omega\beta\sin\omega\tau_0)\cos\gamma(\tau_1 - \tau_0) - \gamma(x_0 - \beta\cos\omega\tau_0)\sin\gamma(\tau_1 - \tau_0) - \\ &\quad - \beta\,\omega\sin\omega\tau_1 \end{aligned} \tag{2.3}$$

$$y_1 = \begin{cases} +d & \text{for } S_{++}, \\ -d & \text{for } S_{--}, \end{cases} \qquad \beta = \frac{1}{1 - (1+\mu)\,\omega^2}, \qquad \gamma = \frac{1}{\sqrt{1+\mu}} \tag{2.4}$$

The value $\tau_1 > \tau_0$ and is the smallest root of the equation $x_1^{\cdot\cdot} = 0$, or

$$-x_1 + \cos\omega\tau_1 = 0 \tag{2.5}$$

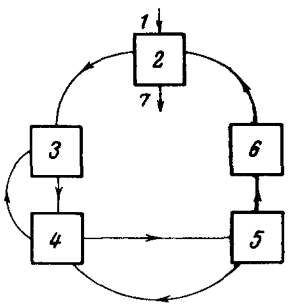

FIG. 2

The initial points of the transformations S_{++} and S_{--} are located on the surfaces $y = + d$ and $y = - d$, respectively. The transformation is applicable after a perfectly inelastic impact ($R = 0$) or after an impact with zero relative velocity if the inequality

$$y_0 y_0^{\cdot\cdot} > 0 \tag{2.6}$$

is satisfied. If this condition is not satisfied, the representative point leaves the collision surface and a transformation of T-type applies.

3. Results of the solution of the problem. The equations of the point transformations (2.1) to (2.3) and (2.5), completely determine the motion of the system if the values of the parameters and the initial position of the representative point in phase space are given.

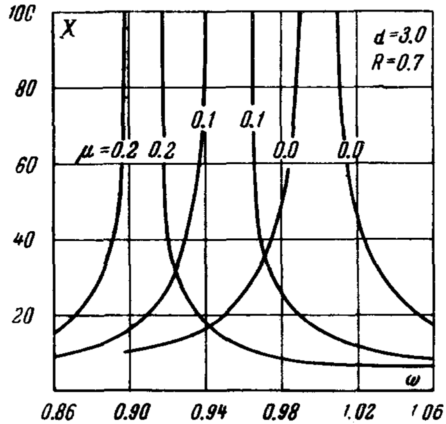

FIG. 3

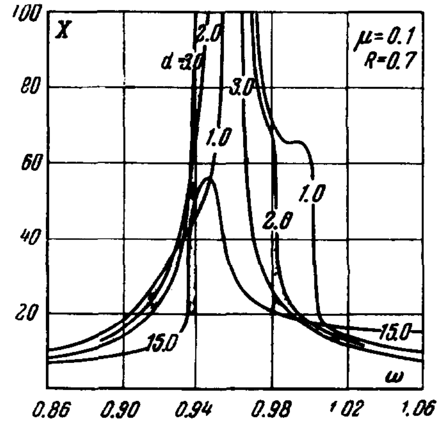

FIG. 4

The study of the resonance behavior of the system under consideration was carried out on the 'Razdan – 2' digital computer of the Gorky Institute of Water Transport Engineers. A flow diagram of the solution is shown in Fig. 2, where *1* is the start; *2* is the specification of the initial conditions and parameters; *3* is the determination of the type of the next transformation; *4* is the calculation of the part of the phase trajectory; *5* is the analysis of the solution; *6* is the printing of the results; and *7* is the stop. The initial position of the representative point was always given at the origin of coordinates. During the solution the maximum excursions $X^i = |x(\tau_i)|$ max were stored and analyzed. Resonance behavior of the system for each parameter value was characterized by the magnitude $X = X^i_{\max}$. The continuation of a sequence of values X^j ceased when one of the following three conditions was fulfilled: (*a*) none of the last N values of X^j turned out to be larger than the previously identified $X^i_{\max}$; (*b*) the value of X exceeded some given magnitude X^*; (*c*) the time τ in the solution exceeded some given value τ^*.

In the solution N was taken as equal to 20, $X^* = 100$, and $\tau^* = 100\ (2\pi/\omega)$. Naturally,

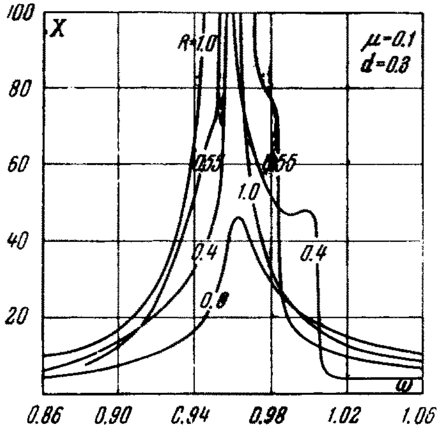

FIG. 5

these conditions on the determination of values of the parameters producing large vibrations may reduce the critical ranges somewhat. Resonance curves for the system are shown in Figs. 3 to 5 in the form of families of relation $X(\omega)$. We note some peculiarities of the forced vibrations of the dynamical system under investigation.

1. If the parameters μ, d, and R are fixed and only the frequency of the external force is changed, the behavior of the system is analogous to that of a system with a single degree of freedom. An increase of the relative mass μ of the impacting body shifts the resonance region toward the lower frequencies Fig. 3).

2. The nonlinearity of the system becomes specially apparent when the relative clearance d and the coefficient of restitution R are varied. The proportionality of the amplitude of forced vibrations to the amplitude of the applied force which holds in linear systems is taken into account in the formulation of the present solution by means of the relation (1.4), $\xi = x\,(F/k)$, between the dimensional and the dimensionless coordinate. The presence of an additional dependence of X on $d = D(k/F)$ (Fig. 4) shows that in the system with collisions the increase of the 'amplitude' of the vibrations may greatly exceed the increase of the amplitude of the external force.

3. The effect of the parameter R on the dissipative properties of the system is complicated, since the energy dissipated in an impact depends greatly on the character of the impact as well as on R. As is clear from Fig. 5, a decrease in R may lead to a sharp increase in the oscillations.

4. The model which is considered here has been studied several times from the point of view of damping by collisions [1, 2 and 3]. In these studies a two-collision-per-cycle, symmetric, periodic motion was investigated, for which the acceleration damper turned out to be very effective. The investigation which has been carried out here shows that the effect of using an acceleration damper may be just the opposite of what is derired if another motion is set up in the system.

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